

Dynamics of correlations in shallow optical lattices

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We explore how correlations evolve in a gas of lattice bosons when the lattice depth is rapidly reduced. We find a simple closed form expression for the static structure factor in the limit of vanishing interactions. The corresponding real-space density correlation function shows multiple spatial oscillations which linearly disperse as a function of time. By perturbatively including the effects of the interactions we calculate how the boson quasi-momentum evolves following the quench.

Introduction.— While many phenomena in nature can be qualitatively understood by simple “mean field”-type theories, these approaches inevitably capture only a subset of the interesting physics. For example, the Mott insulating state of lattice bosons is not inert, rather there is a gas of particle-hole pairs which give rise to a finite correlation length. Similarly, a superfluid is not fully characterized by a coherent state. Recent experimental advances in ultra-cold gases have given us new tools for studying these fluctuations [1–8]. A particularly promising technique (largely unique to cold atoms) is to rapidly change the Hamiltonian parameters such as hopping rate J and interaction strength U . The evolution following such a quench gives many insights into the single and many-particle properties of the system — the spectrum of excitations [9], the manner in which correlations develop [10], and the role of quantum coherence [11]. Here we calculate how various correlation functions evolve after a sudden quench from a strongly interacting Mott insulator ($U \gg J$) to a weakly interacting superfluid ($J \gg U$). Our calculations are closely related to experiments at Munich [10], but we consider a quench to much weaker interactions.

We calculate how density-correlations evolve following a quench, and how quasimomentum is redistributed. By working in the weakly interacting limit we are able to produce analytic expressions. In particular, the time dependence of the static structure factor is quite simple. Our studies complement classical field calculations [12] and sophisticated numerically exact approaches, largely restricted to one-dimension [13–15], or small systems [16]. Remarkably, much of the physics seen in the strongly interacting regime already appears in our calculations. For example, we find that the density correlations spread ballistically [10, 17], and display damped oscillations.

Formalism.— We consider a d -dimensional homogeneous gas of bosons in an cubic optical lattice described by the single-band Bose-Hubbard Hamiltonian [18, 19]:

$$\mathcal{H} = -J \sum_{\langle ij \rangle} (a_i^\dagger a_j + h.c.) + \sum_i \left[\frac{U}{2} n_i (n_i - 1) - \mu n_i \right] \quad (1)$$

where $a_i(\mathbf{t})$ denotes the boson annihilation operator at site i , J denotes the hopping and U the on-site repulsive interaction. The kinetic energy sum is over nearest

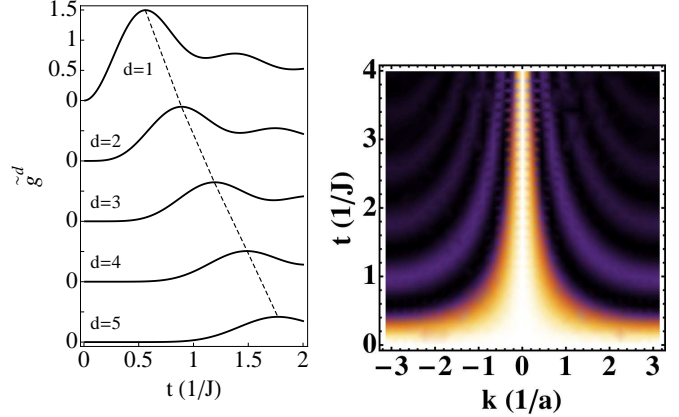


FIG. 1: (color online) **Light-cone evolution of density-density correlations in 1D.** (c.f. Fig. 2, Ref. [10]). Left: Density-density correlations function for a homogeneous, non-interacting system $\tilde{g}_{ji}^{ij}(\mathbf{t}) \equiv \tilde{g}^d(\mathbf{t})$ ($d = i - j$) obtained from Eq. (5). Line shows the location of the peak in g^d used to extract the velocity of spread of correlations. We find $v = 3.7Ja$, consistent with the spreading velocity expected for non-interacting particles. In the experiment of Ref. ([10]) finds a velocity of $v \sim 5.5Ja$, in the strongly interacting limit. Right: Time-Evolution of the structure factor. Lighter colors indicate higher intensity. At $t = 0$, all momenta are equally occupied and $S(k)(0) = 1$ for all k . At intermediate times $S(k)$ shows oscillations due to constructive interference of the atomic wave-packets when they travel a lattice unit. Higher momentum contributions to $S(k)$ decay as $1/Jt$, consistent with the linear spreading of correlations in real-space.

neighbor pairs $\langle ij \rangle$.

The basic objects of our study are the one and two body density matrices; $g_j^i(\mathbf{t}) = \frac{1}{t} \langle a_i^\dagger(\mathbf{t}) a_j(\mathbf{t}) \rangle$ and $g_{kl}^{ij}(\mathbf{t}) = -\langle a_i^\dagger(\mathbf{t}) a_j^\dagger(\mathbf{t}) a_k(\mathbf{t}) a_l(\mathbf{t}) \rangle$, where $a_i(\mathbf{t})$ denotes the boson annihilation operator at site i and time \mathbf{t} . More generally we write the n body density matrix as $g_{j_1 \dots j_n}^{i_1 \dots i_n}(\mathbf{t}) = \frac{1}{n!} \langle a_{i_1}^\dagger(\mathbf{t}) \dots a_{i_n}^\dagger(\mathbf{t}) a_{j_n}(\mathbf{t}) \dots a_{j_1}(\mathbf{t}) \rangle$. In various references, these are also referred to as the n body correlation functions, the $2n$ point functions, or the equal time Green’s functions.

The one and two body correlation functions are readily measured in cold-atom experiments. The former is related to the momentum distribution function $g(\mathbf{k}) = \frac{1}{t} \langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \rangle = \sum_{i,j} e^{i\mathbf{k} \cdot (\mathbf{r}_i - \mathbf{r}_j)} g_j^i$, which is probed through bandmapping [3, 4] or time-of-flight [11]. The density-density correlation function g_{ji}^{ij} is related to the struc-

ture factor $S(\mathbf{q}) = \langle \rho_{\mathbf{q}}^\dagger \rho_{-\mathbf{q}} \rangle = -\sum_{ij} e^{i\mathbf{q} \cdot (\mathbf{r}_i - \mathbf{r}_j)} g_{ji}^{ij}$, where $\rho_{\mathbf{q}} = \sum_{\mathbf{k}} a_{\mathbf{k}+\mathbf{q}}^\dagger a_{\mathbf{k}}$. The structure factor can be probed experimentally with Bragg scattering [1, 2], or sophisticated imaging techniques [6–8].

The equations of motion for the n -body Green's functions are constructed from the equations of motion for the operators $a_i(\mathbf{t})$ and $a_i^\dagger(\mathbf{t})$:

$$i\partial_{\mathbf{t}} a_i = -J(a_{i+1} + a_{i-1}) + U(1 - a_i^\dagger a_i) a_i - \mu a_i \quad (2)$$

where all temporal dependence is implicit.

For the one and two body Green's functions we obtain:

$$i\partial_{\mathbf{t}} g_j^i = -J(g_{j+\langle j \rangle}^i - g_j^{i+\langle i \rangle}) + iU(g_{ij}^{ii} - g_{jj}^{ij}) \quad (3)$$

$$i\partial_{\mathbf{t}} g_{kl}^{ij} = -J(g_{k+\langle k \rangle}^{ij} + g_{kl+\langle l \rangle}^{ij} - g_{kl}^{i+\langle i \rangle j} - g_{kl}^{ij+\langle j \rangle}) + iU(g_{ikl}^{iij} + g_{jkl}^{ijj} - g_{kkl}^{ijk} - g_{kll}^{ijl}) \quad (4)$$

where the notation $\langle i \rangle$ denotes a sum over all the nearest neighbors of site i in d -dimensions. For example, in one dimension $g_{j+\langle j \rangle}^i = g_{j+1}^i + g_{j-1}^i$. In a translationally invariant system (such as the one we consider) $g_{j+\langle j \rangle}^i = g_j^{i+\langle i \rangle}$, and the term proportional to J in Eq. (3) vanishes.

The interaction term couples the n -body Green's function with the $n+1$ -body Green's function. The full interacting many body dynamics is described by the resulting infinite set of coupled differential equations.

Here we limit ourselves for the case of a shallow lattice, where interactions are weak following the quench. We first set $U = 0$ and calculate the *non-interacting* density-density correlation functions (Eq. (4)). We then perturbatively include the effects of U , determining how interactions influence quasi-momentum redistribution in the lattice (Eq. (3)).

Density-Density Correlations in 1D.— We first consider a one-dimensional system as was studied in the experiment of Cheneau *et al.* [10]. We choose a homogeneous initial state with a density of n_0 bosons per site. At $\mathbf{t} = 0$, the sites are completely decoupled, leading to a uniform quasi-momentum distribution with magnitude $g(k) = n_0$.

In the absence of interactions, there is no quasi-momentum redistribution, and the momentum occupations do not change in time. This can be easily seen by taking the Fourier transform Eq. (3). However density-density correlations given by Eq. (4) show interesting dynamics.

To solve Eq. (4), it is convenient to define a reduced two body Green's function $\tilde{g}_{kl}^{ij} = g_{kl}^{ij} - g_k^i g_l^j - g_l^i g_k^j$. As the two body correlations do not evolve in time, \tilde{g}_{kl}^{ij} obeys the same dynamical equation as g_{kl}^{ij} . At $\mathbf{t} = 0$, $\tilde{g}_{kl}^{ij} = n(n+1)$ if $i = j = k = l$, but 0 otherwise. In Fourier space this becomes $\tilde{g}_{rs}^{pq} = n_0(n_0+1)\delta(p+q-r-s)$.

Setting $U = 0$, Eq. (4) is readily solved in Fourier space to yield $\tilde{g}_{rs}^{pq}(\mathbf{t}) = e^{-i2J\mathbf{t}(\cos(p)+\cos(q)-\cos(r)-\cos(s))} g_{rs}^{pq}(\mathbf{t} = 0)$. In real space, the density-density correlation function \tilde{g}_{ji}^{ij} reads:

$$\tilde{g}_{ji}^{ij}(\mathbf{t}) \equiv \tilde{g}^d = n_0(n_0+1) \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{2ikd/a} J_0[4J\mathbf{t} \sin(k)]^2 \quad (5)$$

where $d = i - j$, and $J_\nu(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(\nu k - z \sin(k))} dk$ is the zeroth order Bessel function of first kind.

In Fig. (1), we plot the dynamics of the two body Green's function as a function of time. As is apparent in the figure, the density correlations spread in a light-cone-like manner as observed in Ref. [10]. One can extract a characteristic velocity associated with the ballistic spread of correlations by plotting the location of the maximum of \tilde{g}^d (indicated in Fig. 1 by the dashed line) as a function of d . We obtain a velocity of $v = 3.7Ja$. Note that there are other ways to extract a velocity from this data, which yield somewhat different values.

Given that density correlations result from interference between left and right moving particles, one expects the correlations to spread at roughly twice the maximum group velocity of a wavepacket $v \sim 2v_{\max}$. Indeed, the dispersion $\epsilon(k) = -2J \cos(ka)$ of this system yields $2v_{\max} = 4Ja$ [17, 20]. The correlations spread about 10% slower than this, presumably due to contributions from other parts of the Brillouin zone.

There is a striking similarity between the simple analytic form derived in Eq. (5) for the density-density correlations in a *non-interacting* system and those observed experimentally and numerically in the strongly interacting regime [10, 20]. This similarity is due to the fact that as long as the system is quenched sufficiently far from the critical point of the superfluid-insulator transition, it can be viewed as a dilute gas of weakly interacting quasi-particles, which propagate ballistically. One expects repulsive interactions to alter the propagation velocity of correlations, but have little effect on the functional form of the correlation function itself.

In Fig. (1), we plot the structure factor obtained by taking the Fourier transform of the density-density correlation function. This can simply be read off from Eq. (5) as $S(q)(\mathbf{t}) = n_0(n_0+1)J_0[4J\mathbf{t} \sin(q/2)]^2$. At $\mathbf{t} = 0$, the structure factor is a constant as all momentum states are equally occupied. As the system begins to develop correlations between neighboring sites, the structure factor shows periodic oscillations whose amplitude decays in time. The oscillations indicate a rephasing of the spreading wave-packets as they travel a lattice unit. Using the asymptotic behavior of the Bessel function $J_0(z) \sim (2/\pi z)^{1/2} \cos(z - \pi/4)$ as $z \rightarrow \infty$, we find that for long times the oscillations have period $\tau_{osc} = \pi/[4J \sin(k/2)]$. The envelope of $S(q)$ falls off as $1/J\mathbf{t}$.

A related experiment at MIT studied the Bragg in-

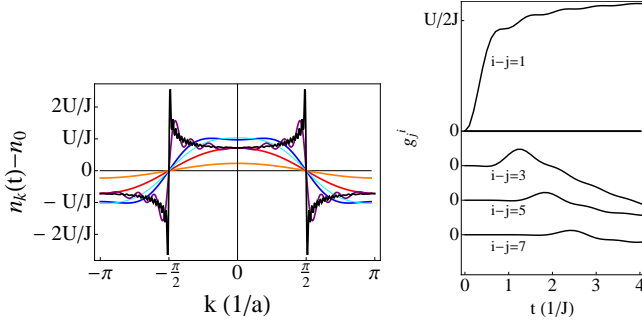


FIG. 2: (color online) **Coherent redistribution of quasi-momentum to linear order in U/J in 1D.** Left: Redistribution of quasi-momentum ($n_k = \langle a_k^\dagger a_k \rangle$) at times $\mathbf{t} = 0$ (thin, black), $\mathbf{t} = 0.25/J$ (orange), $0.5/J$ (red), $0.75/J$ (cyan), $1/J$ (blue), $5/J$ (purple) and $25/J$ (solid, black) obtained by integrating Eq. (6). At short times, we find a coherent transfer of quasi-momentum from high momentum states to low momentum states. At longer times, we find a pile-up of particles near $k = \pi/2$. Right: Spatial evolution of the one-body density matrix $g_j^i(\mathbf{t})$ for different values of $d = i - j$. The correlations vanish if d is even.

tensity of light reflected from a 3D cloud of expanding bosonic atoms released from an optical lattice [2]. Relatively small changes in their experiment would allow them to study the time evolution of the structure factor in a lattice.

Momentum distribution in 1D.— Following the quench we expect interactions to drive the system towards equilibrium. In particular, occupation of low momentum states should grow. To investigate this process we solve Eq. (3), replacing the two body correlator $g_{ij}^{kl}(\mathbf{t})$ with the noninteracting result in Eq. (5). Defining $x \equiv 2J\mathbf{t}$ and using the integral identity $J_\nu(z) = \frac{i^{-\nu}}{2\pi} \int_{-\pi}^{\pi} d\theta e^{i\nu\theta + z \cos(\theta)}$ [21], we find that the occupation numbers obey:

$$\partial_{\mathbf{t}} g_q = U n_0 (n_0 + 1) \sum_{k=-\infty}^{\infty} i^k J_k(-x) J_k(x) \times \quad (6)$$

$$\left(e^{-i(qk - x \cos(q))} J_k(-x) - e^{i(qk - x \cos(q))} J_k(x) \right)$$

The right-hand-side of Eq. (6) is invariant under the inversion $q \rightarrow -q$ but switches sign under the transformation $q \rightarrow \pi - q$. This implies that $q = \pm\pi/2$ is a stationary point and states at $q = \pi/2$ have no dynamics. In real space, this symmetry implies $g_j^i = 0$ if $d = |i - j|$ is even.

In Fig. (2) we plot the evolution of the quasi-momentum states obtained by integrating Eq. (6). At $\mathbf{t} = 0$, all momentum states are equally occupied. At short times following the quench, quasi-momentum states explore the band and the low momentum occupation begins to grow. Peaks develop near the stagnation points $q = \pm\pi/2$. The height of the peak scales as

$U n_0 (n_0 + 1) \sqrt{\mathbf{t}/J}$. Due to this sharp spectral feature, in real space all g_j^i (with $d = |i - j|$ odd) are appreciable at long times.

The structure near $q = \pm\pi/2$ in Fig. 2 are reminiscent of peaks seen in simulations of freely evolving 1D hard core lattice bosons by Rigol and Muramatsu [14]. Despite the similarities, the relationship to our results is not clear, as we have no trap, and only weak interactions. Furthermore, they see a symmetric peak, while ours is antisymmetric.

As shown in Fig. (2), at short times the single particle correlations spread in a manner similar to the density correlations. Local correlations are established on a time of order J^{-1} . Long range order, however, requires communication between widely separated sites. Infinite range order is not found at any finite time.

The linear approximation developed here breaks down whenever the height of the peaks near $q = \pm\pi/2$ become comparable to the initial density. At this point, we expect non-linear processes such as collisions between quasi-momentum states at q and $-q$. This condition, $(n_{q=\pi/2}(\mathbf{t}) - n_0)/n_0 \sim 1$, occurs at time $\mathbf{t} \sim J/(U(n_0 + 1))^2$. We do not know how robust the peaks are against collisions. One could argue that since two-body collisions do not change momentum states in 1D, the quasi-condensate peaks may be long lived.

Single site resolved experiments [13] have studied the dynamics of the nearest neighbor correlation function following a quench to shallow lattices ($V_{\text{final}} \leq 2E_R$). They found that the nearest-neighbor single particles correlations scale linearly with U/J .

Two dimensions.— We now generalize our results to higher dimensions. Concretely, we consider the case of a two-dimensional square lattice, initially containing n_0 particles per site, and investigate the dynamics following a sudden reduction of the lattice depth to the weakly interacting limit.

Repeating our previous 1D arguments in two dimensions, we find that the structure factor is $S(q_x, q_y)(\mathbf{t}) = n_0(n_0 + 1) J_0[4J \mathbf{t} \sin(q_x/2)]^2 J_0[4J \mathbf{t} \sin(q_y/2)]^2$, which decays as $1/\mathbf{t}^2$ (as opposed to the $1/\mathbf{t}$ decay in one-dimension). This power law is a consequence of the ballistic spread of correlations. After a time \mathbf{t} , particle-hole correlations spread over a volume $\sim v^2 \mathbf{t}^2$ where v is twice the characteristic velocity of an atom ($v \sim 4\sqrt{2}Ja$).

The time evolution of the momentum distribution plotted in Fig. (3) is quite different from the results in 1D. Although the analog of Eq. (6) has four stationary points, $\{k_x, k_y\} = \pm\{\pi/2, \pi/2\}$, these become zeros in δn_k rather than peaks. At long times the distribution is characterized by a broad peak centered around $k = 0$. Thus to linear order in U , there is no condensation (characterized by a macroscopic occupation of the $k = 0$ mode) nor quasi-condensation (characterized by a power law divergence of n_k near $k = 0$).

Conclusions and Outlook.— By considering the dy-

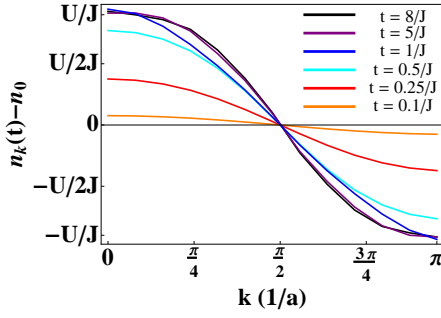


FIG. 3: **Rapid equilibration of momentum distribution in two-dimensions** Left: Momentum distribution along the $\{\pi, \pi\}$ vector obtained by integrating Eq. (3) in two-dimensions assuming an initially uniform distribution. In contrast to the one-dimensional case, the distribution evolves rapidly to a broad peak at $k = 0$, with no further dynamics.

namics of lattice bosons following a quench to a weakly interacting final state, we have explored how correlations develop in a many-body system. Remarkably we find that much of the behavior seen in the strongly interacting system is already present for weak interactions. For example, we find temporal oscillations and decay in the density correlations, similar to those seen in experiments [10]. Within this framework we study how quasi-momentum states evolve following the quench, and investigate the initial stages of condensate formation. Our methods can readily be generalized to investigate fermionic systems, or more complicated lattice geometries, such as triangular and hexagonal lattices, that are at the frontier of cold-atom research [22].

A major open question in the field of non-equilibrium dynamics is how off-diagonal long range order is generated following a quench from a Mott insulating state. Over a decade ago there was a large body of work asking analogous questions with thermal quenches [23]. The picture they developed was one of nucleation and subsequent coarsening. Similar physics is expected in the quantum case [24]. Experimental studies quenching into the regime $\hbar/U \sim \hbar/J \sim 1$ ms find that in 3D a condensate develops on a timescale of order milliseconds [25]. A key ingredient in the equilibration process is collisions, which first appear at order U^2 .

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